Linear Algebra & Geometry

LECTURE 11

- Systems of Linear Equations
- Kronecker-Cappelli Theorem
- Determinant

SYSTEMS OF LINEAR EQUATIONS

A system of linear equations

$$(*)\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \ \dots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \ \dots + a_{2,n}x_n = b_2 \\ \qquad\qquad \cdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \ \dots + a_{m,n}x_n = b_m \end{cases}$$

can be represented as a single matrix equation $AX = B$, where A = $[a_{i,j}]$ is called the coefficient matrix,

$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$. X and B are single-column matrices.

The system of linear equations $(*)$ can also be represented as a vector equation

$$x_1 \underbrace{\begin{bmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{bmatrix}}_{C_1} + x_2 \underbrace{\begin{bmatrix} a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{m,2} \end{bmatrix}}_{C_2} + \ \ldots + x_n \underbrace{\begin{bmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m,n} \end{bmatrix}}_{C_n} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{B}$$

Which means we are trying to find coefficients to express B as a linear combination of columns of A. This can only be done if

$$\text{span}\{C_1, C_2, \ldots, C_n\} = \text{span}\{C_1, C_2, \ldots, C_n, B\}.$$

The matrix with columns $C_1, C_2, \ldots, C_n$ and B is called *the augmented matrix* of the system of equations and is denoted by $[A|B]$.

Theorem. (Kronecker, Cappelli)

A system $AX = B$ of linear equations has a solution iff

$$r(A) = r([A|B]).$$

Proof. The vector-oriented approach form the previous slide together with properties of the *span* operation is proof enough.

Remark.

Interchanging equations, multiplying both sides by a non-zero number and adding equations one to another do not affect the set of solutions of a system of equations. EROS are exactly these operations except that they are performed on rows of a matrix rather than on equations. This suggests a strategy for solving a system of equations. Start with a system $(*)$, represent it as its augmented matrix [A|B], row-reduce the matrix to a row echelon matrix [E|C], translate the matrix to the language of equations.

Consider the system

$$\begin{cases} 2x + 4y - z = 11 \\ -4x - 3y + 3z = -20 \\ 2x + 4y + 2z = 2 \end{cases}$$

Its augmented matrix is

$$\begin{bmatrix} 2 & 4 & -1 & 11 \\ -4 & -3 & 3 & -20 \\ 2 & 4 & 2 & 2 \end{bmatrix} \sim (r_3 - r_1, r_2 + 2r_1) \begin{bmatrix} 2 & 4 & -1 & 11 \\ 0 & 5 & 1 & 2 \\ 0 & 0 & 3 & -9 \end{bmatrix}.$$

Clearly, the rank of both A and $[A|B]$ is 3 which means the system is solvable. Let us reduce $[A|B]$ to a row-canonical matrix.

$$\sim (r_1 + \tfrac{1}{3}r_3, r_2 - \tfrac{1}{3}r_3, \tfrac{1}{3}r_3) \begin{bmatrix} 2 & 4 & 0 & 8 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 1 & -3 \end{bmatrix} \sim \left(r_1 - \tfrac{4}{5}r_2, \tfrac{1}{5}r_2\right)$$

$$\begin{bmatrix} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{bmatrix} \sim (\tfrac{1}{2}r_1) \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{bmatrix},$$ which is the matrix of

$x = 2, y = 1\ and\ z = -3.$

Definition.

A system of linear equations $AX = B$ is called *homogeneous* iff $B = \Theta$.

Fact.

Every homogeneous system of linear equations has a solution, namely $x_1 = 0, x_2 = 0, \ldots, x_n = 0$. Any other solution (if there is one) is called a *non-trivial* or *non-zero* solution.

Theorem.

Let $AX = \Theta$ be a homogeneous system of m linear equations with n unknowns. Then the set $W = \{v \in \mathbb{K}^n | Av = \Theta\}$ of all solutions of the system is a subspace of the vector space $\mathbb{K}^n$. Moreover,

$\dim(W) = n - r(A)$.

Proof. (of the first statement)

Take $u, v \in W$. This means that $Au = \Theta$ and $Av = \Theta$. Since matrix multiplication is distributive over addition, we have $A(u + v) = Au + Av = \Theta + \Theta = \Theta$ i.e., $u + v \in W$.

Similarly, we prove that for every $k \in \mathbb{K}$ we have $A(ku) = k(Au) = k\Theta = \Theta$.

We skip the proof of the second statement. QED

Example.

$$\begin{cases} x + y - z = 0 \\ 2x - 3y + z = 0 \\ x - 4y + 2z = 0 \end{cases} \quad A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 1 \\ 1 & -4 & 2 \end{bmatrix} \sim r_2 - 2r_1, r_3 - r_1 \sim$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 3 \\ 0 & -5 & 3 \end{bmatrix} \sim r_3 - r_2 \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

. The rank of the last matrix is 2. Hence the dimension of the solution space is $3 - 2 = 1$.

We shall find a basis for the space reducing the matrix further.

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 3 \\ 0 & 0 & 0 \end{bmatrix} \sim \frac{1}{-5} r_2 \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & \frac{-3}{5} \\ 0 & 0 & 0 \end{bmatrix} \sim r_1\text{-}r_2 \sim \begin{bmatrix} 1 & 0 & \frac{2}{5} \\ 0 & 1 & \frac{-3}{5} \\ 0 & 0 & 0 \end{bmatrix}.$$

In the language of equations this reads

$$\begin{cases} x + \qquad \frac{2}{5}z = 0 \\ \qquad y - \frac{3}{5}z = 0 \\ \qquad\qquad 0z = 0 \end{cases}.$$

The bottom equation really says, "z may be anything you like" and the top two say $x = -\frac{2}{5}z$ and $y = \frac{3}{5}z$. Hence every vector (x, y, z) belonging to the solution space looks like $(-\frac{2}{5}z, \frac{3}{5}z, z) = z(-\frac{2}{5}, \frac{3}{5}, 1)$ and the set $\{(-\frac{2}{5}, \frac{3}{5}, 1)\}$ is a one-element basis for the space.

Example.

$$\begin{cases} x + ay + az = 1 \\ ax + ay + z = 1 \\ ax + y + az = 1 \\ ax + ay + az = 1 \end{cases}$$

Discuss solvability of the system in terms of a.

$$\begin{bmatrix} 1 & a & a & 1 \\ a & a & 1 & 1 \\ a & 1 & a & 1 \\ a & a & a & 1 \end{bmatrix} \sim \begin{matrix} r_2 - ar_1 \\ r_3 - ar_1 \\ r_4 - ar_1 \end{matrix} \sim \begin{bmatrix} 1 & a & a & 1 \\ 0 & a - a^2 & 1 - a^2 & 1 - a \\ 0 & 1 - a^2 & a - a^2 & 1 - a \\ 0 & a - a^2 & a - a^2 & 1 - a \end{bmatrix}$$

. Since for a=1 everything except the top row becomes nil, it looks like a good idea to split cases.

Case 1, a=1. We get
$$\begin{bmatrix} 1 & a & a & 1 \\ 0 & a - a^2 & 1 - a^2 & 1 - a \\ 0 & 1 - a^2 & a - a^2 & 1 - a \\ 0 & a - a^2 & a - a^2 & 1 - a \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In this case $r(A) = r(A|B) = 1$. This means the system is solvable. It is reduced to $x + y + z = 1$ hence, $x = 1 - y - z$; y and z are free.

Case 2, $a \neq 1$.

$$\begin{bmatrix} 1 & a & a & 1 \\ 0 & a-a^2 & 1-a^2 & 1-a \\ 0 & 1-a^2 & a-a^2 & 1-a \\ 0 & a-a^2 & a-a^2 & 1-a \end{bmatrix} \sim \textit{divide rows } 2{,}3, \textit{ and } 4 \textit{ by} (1-a) \sim$$

$$\begin{bmatrix} 1 & a & a & 1 \\ 0 & a & 1+a & 1 \\ 0 & 1+a & a & 1 \\ 0 & a & a & 1 \end{bmatrix} \sim \textit{subtract row 4 from other rows} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a & a & 1 \end{bmatrix} \sim r_2 \leftrightarrow r_3 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & a & a & 1 \end{bmatrix} \sim r_4 - ar_2 - ar_3 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$r(A) = 3, r(A|B) = 4$. The system is inconsistent.

Theorem.

Let $AX = B$ be an arbitrary system of linear equations. Let U be the solution set and let $v_0 \in U$ be a solution to the system.

Then $U = v_0 + W = \{v_0 + w | w \in W\}$, where W is the solution space of the corresponding homogeneous system $AX = \Theta$.

Proof.

Each vector $v = v_0 + w$ from $v_0 + W$ is a solution to $AX = B$. Indeed,

$A(v_0 + w) = Av_0 + Aw = B + \Theta = B$. Hence $v_0 + W \subseteq U$

Moreover, for every vector $z \in U$ we may put $t = z - v_0$ so that $z = v_0 + t$. Then

$At = A(z - v_0) = Az - Av_0 = B - B = \Theta$

which means $t \in W$. Hence, $U \subseteq v_0 + W$. QED

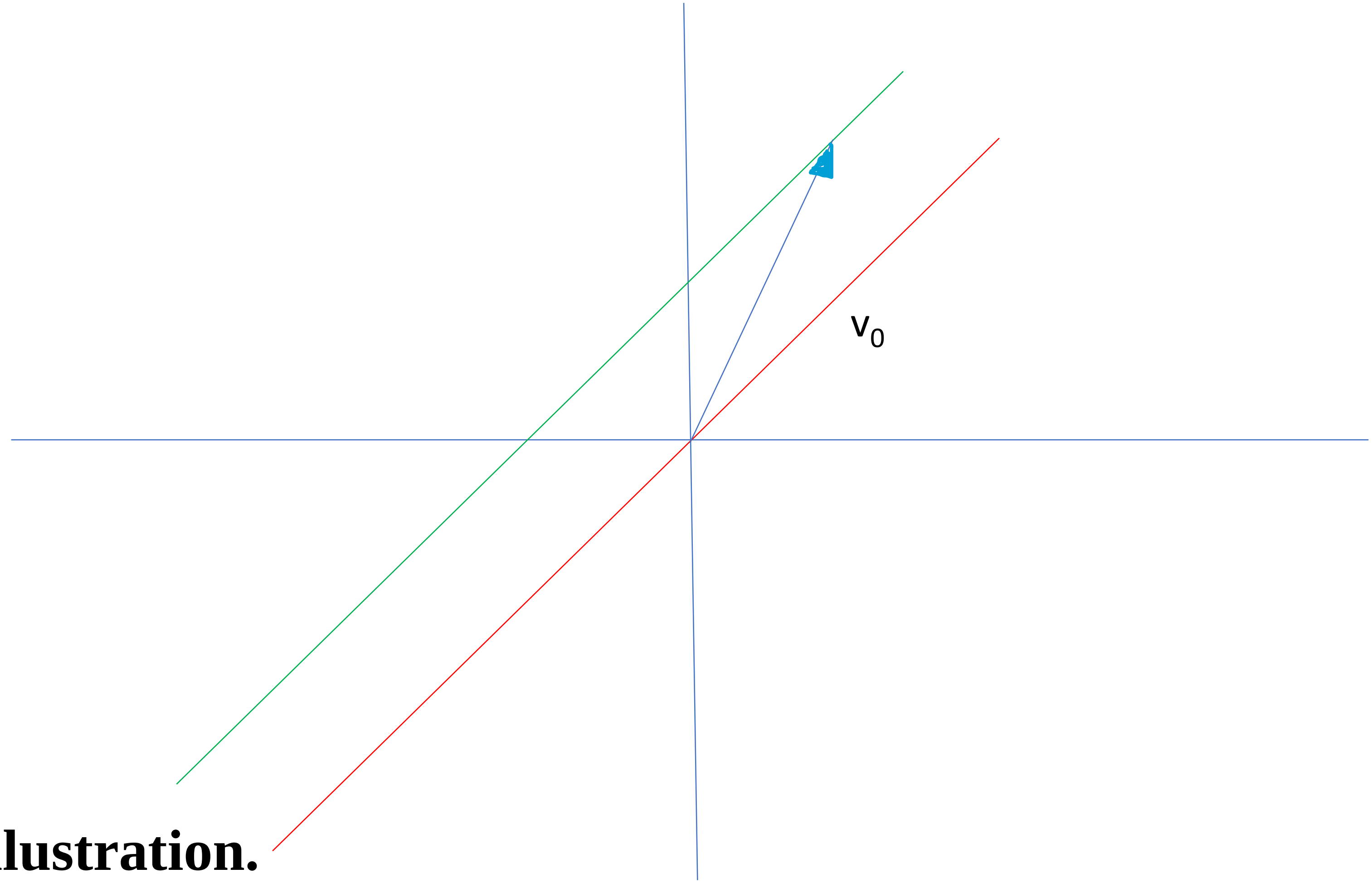


Illustration.

(1) $\{-x + y = 1$ (a system of equation, one equation two unknowns)

(2) $\{-x + y = 0$ (the corresponding homogeneous system)

v_0 – a solution of (1)

Definition.

Determinant (*det* for short) is a function defined on the set of all square matrices ($n \times n$ matrices, n=1,2, …) over a field $\mathbb{K}$ into $\mathbb{K}$. The definition is inductive with respect to n:

1. if $n = 1$, ($A = [a_{1,1}]$) then det(A) = $a_{1,1}$
2. if $n > 1$

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{i,1} \det(A_{i,1})$$

where $A_{i,j}$ denotes the matrix obtained from A by the removal of row number i and column j. det(A) is also denoted by $|A|$.

The formula is known as *Laplace expansion on column 1*.

Notice that we only use the symbol $A_{i,j}$ in the case $j = 1$.

Example.

1. Find $det\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$.

$$\det(A) = \sum_{i=1}^{2} (-1)^{i+1} a_{i,1} \det\left(A_{i,1}\right) = a_{1,1}a_{2,2} - a_{2,1}a_{1,2}$$

In particular, $\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = 2 \cdot (-2) - 1 \cdot 3 = -7$

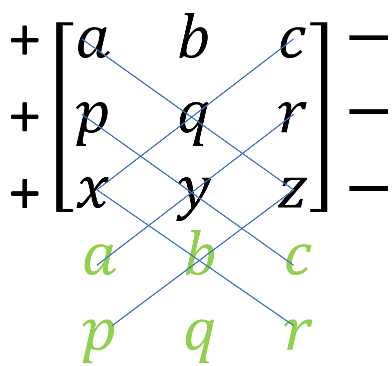
Example.

2. $det\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} = (-1)^{1+1} a\ det\begin{bmatrix} q & r \\ y & z \end{bmatrix} + (-1)^{2+1} p\ det\begin{bmatrix} b & c \\ y & z \end{bmatrix}$

$+ (-1)^{3+1} x\ det\begin{bmatrix} b & c \\ q & r \end{bmatrix} = a(qz - ry) - p(bz - cy) + x(br - qc) =$

$aqz+pyc+xbr-cqx-rya-zbp$. The last formula is known as the *Sarrus Rule.*

BEWARE !. It only works for 3 × 3 matrices.



Theorem.

For every $j = 1,2, \ldots, n$ and for every $n \times n$ matrix A

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} \det(A_{i,j})$$

Proof. Omitted.

Remark. The theorem says that instead of Laplace expansion on column 1 we can do Laplace expansion on column *j*, for every *j*.

Example.

1. Find $det\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$ by *Laplace expansion* on column 2.

$$\det(A) = \sum_{i=1}^{2} (-1)^{i+2} a_{i,2} \det\left(A_{i,2}\right) = -a_{1,2}a_{2,1} + a_{2,2}a_{1,1}$$

Theorem. (determinant versus transposition)

For every matrix A $\det(A) = \det(A^T)$

Proof. Omitted.

Remark. The theorem says (indirectly) that instead of Laplace expansion on columns we can do Laplace expansion on rows.

Example.

1. Find $det\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$ by *Laplace expansion* on row 1.

$$\det(A) = \sum_{i=1}^{2} (-1)^{1+i} a_{1,i} \det\left(A_{1,i}\right) = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$$

Theorem. (*determinant* versus EROS)

For every matrix A

1. If $A \sim (r_i \leftrightarrow r_j)B$ then $\det(B) = -\det(A)$ ($i \neq$ j)
2. If $A \sim (r_i \leftarrow cr_i)B$ then $\det(B) = c\det(A)$
3. If $A \sim (r_i \leftarrow r_i + r_j)B$ and $i \neq$ j then $\det(B) = \det(A)$
4. Combining 3 with 2 we get

   If $A \sim (r_i \leftarrow r_i + cr_j)B$ and $i \neq$ j then $\det(B) = \det(A)$.

Proof. Omitted.

Remark. Thanks to the transposition law the theorem applies also to column rather than row operations.

Remark.
"$A \sim (r_i \leftarrow cr_i)B$" means "$B$ has been obtained from A by replacing r_i of A with cr_i".

Theorem. (determinant versus not-quite-matrix-addition)

Suppose $s \in \{1,2,\ldots,n\}$ and $A[i,j] = B[i,j] = C[i,j]$ for every i,j such that $j \neq s$ and $C[i,s] = A[i,s] + B[i,s]$. Then $\det(C) = \det(A) + \det(B)$.

Proof..

$$\det\begin{bmatrix} c_{1,1} & \cdots & a_{1,s}+b_{1,s} & \cdots & c_{1,n} \\ c_{2,1} & \cdots & a_{2,s}+b_{2,s} & \cdots & c_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{n,1} & \cdots & a_{m,s}+b_{n,s} & \cdots & c_{n,n} \end{bmatrix} =$$

By Laplace expansion on column s

$$\sum_{i=1}^{n}(-1)^{i+s}(a_{i,s}+b_{i,s})\det(C_{i,s}) = \sum_{i=1}^{n}(-1)^{i+s}a_{i,s}\det(C_{i,s}) + \sum_{i=1}^{n}(-1)^{i+s}b_{i,s}\det(C_{i,s}) = \det(A) + \det(B).$$

Warning. This is NOT about determinant of the sum of two matrices being equal to the sum of their determinants; **that is not true.** This is about determinant of a matrix whose ONE column is the sum of two vectors.